Let the sequence $u_n$ with $n$ starting at 0 be defiend as

$$u_n = 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \cdots \underbrace{k, k, k, \cdots, k}_{k \text{ times}}, \cdots$$

The task is to find an equation for $u_n$ in terms of $n$, making use of the 'floor' function, where $\lfloor x \rfloor$ is defined as the largest integer less than or equal to $x$ (ie $x$ rounded down).

Firstly, given an element of the sequence $u_n$, we find upper and lower bounds for $n$. For a given number $u_n$ there are $\frac{1}{2}(u_n - 1)u_n$ numbers in the sequence smaller than $u_n$. This is because the first $1 + 2 + 3 + \cdots + (u_n - 1)$ numbers are all of the numbers smaller than $u_n$. Further, and for the same reason, $u_n$ must be found in the first $\frac{1}{2}u_n(u_n + 1)$ numbers. Hence we write an inequality for $n$:

$$\frac{1}{2}u_n(u_n - 1) < n \leq \frac{1}{2}u_n(u_n + 1)$$

Now we break the inequality into two parts, solving them seperately. Firstly, the lower bound:

$$\begin{aligned} \frac{1}{2}u_n(u_n - 1) &< n \\ u_n^2 - u_n - 2n &< 0 \end{aligned}$$

By solving for the roots of the quadratic upper and lower bounds for $u_n$ can be found:

$$\frac{1 - \sqrt{1 + 8n}}{2} < u_n < \frac{1 + \sqrt{1 + 8n}}{2}$$

We handle the original upper bound in a similar manner:

$$\begin{aligned} n &\leq \frac{1}{2}u_n(u_n + 1) \\ 0 &\leq u_n^2 + u_n - 2n \end{aligned}$$

Once again finding the roots of the right hand side yields two inequalities for $u_n$:

$$u_n \leq \frac{-1 - \sqrt{1 + 8n}}{2} \quad \text{or} \quad u_n \geq \frac{-1 + \sqrt{1 + 8n}}{2}$$

Now these bounds describe a range of values for $u_n$:

$$\frac{-1 + \sqrt{1 + 8n}}{2} \leq u_n < \frac{1 + \sqrt{1 + 8n}}{2}$$

The bounds for this range are $\frac{\sqrt{1+8n}}{2} \pm \frac{1}{2}$. Clearly then, there is only one integer in this range, and by taking the floor of the larger value this integer is obtained. Hence

$$u_n = \left\lfloor \frac{1 + \sqrt{1 + 8n}}{2} \right\rfloor$$